Indian Statistical Institute Bangalore

Statistics and Mathematics Unit

Complex Analysis

M. Math. I Year – Mid-Semester Examination

Instructor: Shubham Jain

Duration: 3 Hours **Total Marks: 50**

Name: ______________________________

Question 1. For each of the following statements, determine whether it is true or false. Justify your answer.

(1) Let f be a holomorphic function in a domain Ω. If the real part of f attains a local minimum at some point in Ω, then f is constant. **[3 Marks]**

(2) Let $w : [0,1] \to \mathbb{C}$ be a continuous function. Then there exists $c \in [0,1]$ such that

$$\int_0^1 w(t)\,dt = w(c).$$

[4 Marks]

(3) There exists a power series that converges exactly on the set

$$\{z \in \mathbb{C} : |z| \leq 1 \text{ and } z^5 \neq 1\}.$$

[5 Marks]

Question 2. Let $f : \Omega \to \mathbb{C}$ be holomorphic, where Ω is a domain. Let γ_1 and γ_2 be two smooth closed curves in Ω that are smoothly homotopic in Ω. Show that

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz.$$

[4 Marks]

Question 3. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of holomorphic functions converging uniformly on compact subsets of a domain Ω. Assume that each f_n has no zeroes in Ω. Show that the limit function f either vanishes identically or has no zeroes in Ω. **[6 Marks]**

Question 4. Compute $\displaystyle\int_C \left(\overline{z}^3 + \overline{z}^7 + \overline{z}\right) dz$, where $C = \{e^{2\pi i t^2} : t \in [0,1]\}$.

[4 Marks]

Question 5. Show that

$$\int_0^\infty \frac{x \sin x}{x^2+1} dx = \frac{\pi}{2e}$$

[8 Marks]

Question 6. Let f be a continuous function on the unit circle

$$C = \{z \in \mathbb{C} : |z| = 1\},$$

with positive orientation.

Define $g : \mathbb{D} \to \mathbb{C}$ by

$$g(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z}\, dw.$$

Show that:

(1) g is holomorphic in $\mathbb{D}$.

(2) **Prove or disprove:** If $f(z) \neq 0$ for all $z \in C$, then there exists $z_0 \in \mathbb{D}$ such that $g(z_0) \neq 0$.

[10 Marks]

Question 7. Let

$$p(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_0$$

be a polynomial, where $a_0, \ldots, a_{n-1} \in \mathbb{C}$.

Using Cauchy's theorem, show that there exists $z_0 \in \mathbb{C}$ such that

$$p(z_0) = \pi.$$

[6 Marks]